# Seshadri fibrations of algebraic surfaces

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**Abstract**

We refine results of [6] and [10] which relate local invariants - Seshadri constants - of ample line bundles on surfaces to the global geometry - fibration structure. We show that the same picture emerges when looking at Seshadri constants measured at any finite subset of the given surface.

## 1   Introduction

Seshadri constants were introduced by Demailly [2] in an attempt to tackle the Fujita Conjecture [4]. They quickly became an object of independent studies. Nakamaye [8] observed that relatively small values of Seshadri constants in a general (and hence every) point of an algebraic surface enforce a fiber space structure on that surface. The same principle was exhibited by Hwang and Keum [6] for varieties of arbitrary dimensions. However only in the case of surfaces, there are some effective bounds. In [10] Tutaj-Gasińska and the second author gave a sharp bound for imposing on a surface a fibration by Seshadri curves. The cubic surface provides an example showing that the obtained bound is in fact optimal.

In the present paper we study the geometry of surfaces a little bit closer. First we show that the cubic surface is the only one for which the bound from [10] is sharp. Hence we are in a position to provide a better bound for all other surfaces.

Secondly we turn to multiple point Seshadri constants. Somehow surprisingly the situation turns out to be similar to that of Seshadri constants in a single point. In a sense it is even better, as with the number of points increasing our bounds converge to the maximal possible value. A similar asymptotic verification of the Nagata-Biran Conjecture (see 4.3) was obtained before with different methods by Harbourne [5, Theorem I.1]. We conclude showing that our bounds are optimal for arbitrary number of points.

## 2   Preliminaries and auxiliary results

In this section we recall basic properties of Seshadri constants and collect some helpful inequalities.

First we recall the following definition. Here $r$ is a positive integer.

**Definition 2.1** *Let $X$ be a smooth projective variety, let $L$ be an ample line bundle on $X$ and let $P_1, \ldots, P_r \in X$ be mutually distinct points. The $r$-tuple Seshadri constant of $L$ at $P_1, \ldots, P_r$ is the real number*

$$\varepsilon(L; P_1, \ldots, P_r) = \inf_{C \cap \{P_1, \ldots, P_r\} \neq \emptyset} \frac{L.C}{\sum \mathrm{mult}_{P_i} C} \,,$$

*where the infimum is taken over all curves passing through at least one of the points*
$P_1, \ldots, P_r$.
*We say that a curve $C$ is a Seshadri curve for the $r$-tuple $P_1, \ldots, P_r$ if*

$$\varepsilon(L; P_1, \ldots, P_r) = \frac{L.C}{\sum \operatorname{mult}_{P_i} C}.$$

Let $f : Y \longrightarrow X$ be the blowing up of $P_1, \ldots, P_r \in X$ with exceptional divisors $E_1, \ldots, E_r$. Equivalently the Seshadri constant can be computed as

$$\varepsilon(L; P_1, \ldots, P_r) = \sup \left\{ \lambda > 0 : f^*L - \lambda \cdot \sum_{i=1}^{r} E_i \text{ is nef} \right\}.$$

Since the self-intersection of a nef line bundle is non-negative, it follows that there is an upper bound:

$$\varepsilon(L; P_1, \ldots, P_r) \leq \sqrt[\dim X]{\frac{L^{\dim X}}{r}} =: \varepsilon_{\text{upper}}(L; r).$$

As a function on $X^r$ Seshadri constant $\varepsilon(L; \cdot, \ldots, \cdot)$ is semi-continuous and has the maximal value at a very general point of $X^r$ i.e. on a subset of $X^r$ which is the complement of a union of at most countably many Zariski closed subsets. We abbreviate this maximal value by $\varepsilon(L; r)$.

**Remark 2.2** *We recall that on a surface $X$ a strict inequality*

$$\varepsilon(L; P_1, \ldots, P_r) < \varepsilon_{\text{upper}}(L; r),$$

*implies via the real valued Nakai-Moishezon criterion [1] that there is a Seshadri curve for the $r$-tuple $P_1, \ldots, P_r$. Such a curve can be assumed to be reduced and irreducible.*
*In particular if*

$$\varepsilon(L; r) < \varepsilon_{\text{upper}}(L; r),$$

*then there is a Seshadri curve through every $r$-tuple of points on $X$.*

The following lemma which is due to Xu [11, Lemma 1] will be used to estimate the self-intersection of Seshadri curves.

**Lemma 2.3** *Let $X$ be a smooth projective surface, let $(C_t, (P_1)_t, \ldots, (P_r)_t)_{t \in T}$ be a non-trivial one parameter family of pointed reduced and irreducible curves on $X$ and let $m_i$ be positive integers such that $\operatorname{mult}_{(P_i)_t} C_t \geq m_i$ for all $i = 1, \ldots, r$. Then*

*for $r = 1$ and $m_1 \geq 2$     $C_t^2 \geq m_1(m_1 - 1) + 1$ and*

*for $r \geq 2$                    $C_t^2 \geq \sum_{i=1}^{r} m_i^2 - \min\{m_1, \ldots, m_r\}$.*

The second lemma was obtained by Küchle in [7] and is purely numerical.

**Lemma 2.4** *Let $r \geq 2$ and $m_1, \ldots, m_r \in \mathbb{Z}$ be integers with $m_1 \geq \ldots \geq m_r \geq 1$ and $m_1 \geq 2$. Then we have*

$$(r + 1) \sum_{i=1}^{r} m_i^2 > \left( \sum_{i=1}^{r} m_i \right)^2 + m_r(r + 1).$$

2

# 3  Single point Seshadri constants and fibrations

Recall that in the case of algebraic surfaces Hwang and Keum proved the following result [6, Theorem 2].

**Theorem 3.1 (Hwang-Keum)** *Let $X$ be a projective surface and $L$ an ample line bundle on $X$ with*

$$\varepsilon(L; 1) < \sqrt{\frac{3}{4}} \cdot \varepsilon_{\text{upper}}(L; 1).$$

*Then there is a fibration of $X$ whose fibers are Seshadri curves of $L$.*

It was shown in [10] that if $X \subset \mathbb{P}^3$ is a smooth cubic surface and $L = \mathcal{O}_X(1)$, then

$$\varepsilon(L; 1) = \sqrt{\frac{3}{4}} \cdot \varepsilon_{\text{upper}}(L; 1) = \frac{3}{2}$$

and that $X$ is not fibered by Seshadri curves. This means that the factor of $\sqrt{\frac{3}{4}}$ in the above cannot be improved in general. Here we show however that the cubic is the only example of this kind.

**Theorem 3.2** *Let $X$ be a smooth projective surface and $L$ an ample line bundle on $X$ such that*

$$\varepsilon(L; 1) = \sqrt{\frac{3}{4}} \cdot \varepsilon_{\text{upper}}(L; 1).$$

*If $X$ is not fibered by Seshadri curves, then $X$ is the cubic surface in $\mathbb{P}^3$ and $L$ is the hyperplane bundle.*

*Proof.* For any $x \in X$ let $D_x$ be an irreducible and reduced Seshadri curve for $x$ (see Remark 2.2) with multiplicity $m_x = \text{mult}_x(D_x)$ in that point. Further, let $X_0 \subset X$ be an open and dense subset of $X$ on which the multiplicity of Seshadri curves is constant equal $m$.

The proof goes in several steps. Here is the outline. First with Hodge Index we show that $m = 2$ (note that for the cubic surface Seshadri constants are computed by tangent sections). In the second step we use Kodaira-Spencer map to prove that the curves $D_x$ are rational. Finally with some ad hoc arguments we conclude that $D_x$ are members of a linear system embedding $X$ as a cubic in $\mathbb{P}^3$.

Let $x \in X_0$ be given. If $m = 1$, then by Hodge Index Theorem we have

$$\frac{3}{4} L^2 = (L.D_x)^2 \geq L^2 D_x^2,$$

which implies $D_x^2 = 0$ and, as in the proof of [10, Theorem], that there is a Seshadri fibration on $X$.

Hence $m \geq 2$ and by Lemma 2.3 we have

$$D_x^2 \geq m(m - 1) + 1.$$

Combining this inequality with our numerical assumptions on $L$ and applying Hodge Index we obtain

$$\frac{3}{4} \geq \frac{m(m - 1) + 1}{m^2},$$

3

which is equivalent to $(\frac{1}{2}m - 1)^2 \leq 0$. This implies $m = 2$ and since then there is an equality in the Hodge Index, we conclude that $D_x^2 = 3$ and $\mathcal{O}_X(D_x)$ is ample, as it is numerically equivalent to some rational multiple of $L$.

Now we turn to the rationality of $D_x$. Since the curves $D_x$ are reduced and $m = 2$ it can happen only for finitely many points $y$ that $D_x = D_y$. This means that we have a two-parameter family of Seshadri curves. We fix $x_0$ in the interior of $X_0$ and a sufficiently small disk $\Delta$ so that $\Delta \times \Delta \subset X_0$ is a neighborhood of $x_0$. Then the deformation $\left( D_{x_{(t,s)}}, x_{(t,s)} \right)_{\Delta \times \Delta}$ of the pointed Seshadri curve $(D_0, x_o)$ determines a non-degenerate Kodaira-Spencer map

$$\rho : T_0\Delta \times T_0\Delta \longrightarrow H^0(D_{x_0}, \mathcal{O}_{D_{x_0}}(D_{x_0})).$$

We abbreviate $D = D_{x_0}$. As in [3, Corollary 1.2] we conclude that there are two independent sections $\rho(\frac{d}{dt})$ and $\rho(\frac{d}{ds})$ in $H^0(D, \mathcal{O}_D(D) \otimes \mathcal{I}_{x_0})$, where $\mathcal{I}_{x_0}$ is the maximal ideal.

Let $f : Y \longrightarrow X$ be the blowing up of $X$ at $x_0$ with the exceptional divisor $E$ and let $D' = f^*D - 2E$ be the proper transform of $D$. By the projection formula we have that $M = f^*\mathcal{O}_D(D) \otimes \mathcal{O}_Y(-E)$ has at least two global sections. On the other hand $\deg M = 1$, which implies that $D'$ is rational and hence so is $D$.

Thus $X$ is rationally connected (any two curves $D_x$ and $D_y$ intersect), so it is a rational surface. In particular all curves $D_x$ are linearly equivalent and equivalent to $L$. We denote the linear system generated by the curves $D_x$ simply by $|D|$. Note that we have thus obtained a complete family of Seshadri curves. Since for a very general point $x \in X$ there exists in $|D|$ a curve singular at $x$, we find such a curve (possibly reducible) for every point on the surface.

Now we show that the linear system $|D|$ is base point free. To this end let $y \in X$ be fixed. There is a curve $D_y \in |D|$ with $\mathrm{mult}_y D_y \geq 2$. Let $y_1$ be a general smooth point of $D_y$. Again, there exists an irreducible curve $D_{y_1} \in |D|$ with $\mathrm{mult}_{y_1} D_{y_1} = 2$. We claim that this curve does not go through y. Indeed, since $D_y$ and $D_{y_1}$ have no common components and their intersection product is 3 they cannot meet each other in singular points.

Taking into account that $|D|$ is ample and base point free, the image of the induced mapping of $X$ has dimension 2. Bertini's theorem tells us that a general member of $|D|$ is smooth and irreducible. Since we have already identified a two-parameter family of singular divisors in $|D|$, it implies that $\dim |D| \geq 3$.

Finally we prove that the linear system $D$ defines an embedding of $X$. First we show that $|D|$ separates points. This goes similarly as the global generation. Let $y_1$ and $y_2$ be two different points on $X$ and let $D_{y_1}$ and $D_{y_2}$ be the corresponding Singular curves in $|D|$. If $y_1 \notin D_{y_2}$, then we are done. If $y_1 \in D_{y_2}$ it cannot be $y_2 \in D_{y_1}$ at the same time, otherwise it contradicts $D_{y_1}.D_{y_2} = 3$, so we are done again.

Now, let $x \in X$ and $\overrightarrow{v} \in T_xX$ be fixed and let $D_x$ be the singular curve at $x$ in $|D|$. If $\overrightarrow{v}$ is not tangent to $D_x$, then we are done. Suppose that all curves in the system $|D \otimes \mathcal{I}_x|$ have $\overrightarrow{v}$ as a tangent vector, then all these curves intersect $D_x$ only in $x$, as the intersection multiplicity already at that point is equal 3 or they have a common component with $D_y$. On the other hand let $x_1 \in D_x$ and $x_2 \notin D_x$ be general. Dimension count shows that there is an irreducible curve

$C \in |D \otimes \mathcal{I}_x \otimes \mathcal{I}_{x_1} \otimes \mathcal{I}_{x_2}|$. Such a curve has no common components with $D_x$ and would have intersection multiplicity $\geq 4$, a contradiction.

Summing up, we have shown that the linear system $|D|$ embeds $X$ as a surface of degree 3 in a projective space. Since a complete embedding is non-degenerate, the image of $X$ must be a smooth cubic in $\mathbb{P}^3$. $\qquad\square$

Now we are in a position to improve the bound in Theorem 3.1.

**Corollary 3.3** *Let $X$ be a smooth projective surface and $L$ an ample line bundle on $X$. If*

$$\varepsilon(L;1) < \sqrt{\frac{7}{9}} \cdot \varepsilon_{\text{upper}}(L;1),$$

*then*

a) *either $X$ is a cubic in $\mathbb{P}^3$ and $L = \mathcal{O}_X(1)$,*

b) *or $X$ is fibered by Seshadri curves.*

*Proof.* We assume to the contrary that $X$ is neither a cubic nor is it fibered by Seshadri curves. From the proof of Theorem 3.2 it follows immediately that the multiplicity of Seshadri curve at a general point of $X$ is at least 3. On the other hand combining Lemma 2.3 with the Hodge Index and our numerical assumption, it is elementary to see that this is impossible. $\qquad\square$

# 4 Multiple point Seshadri constants and fibrations

We now pass to the second part of our paper and investigate the relationship between multiple point Seshadri constants and fibrations by Seshadri curves.

**Theorem 4.1** *Let $X$ be a smooth projective surface, $L$ an ample line bundle on $X$ and $r \geq 2$ a fixed integer. If*

$$\varepsilon(L;r) < \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\text{upper}}(L;r) \tag{1}$$

*then there exists a fibration $f : X \longrightarrow D$ over a curve $D$ such that given $P_1, \ldots, P_r \in X$ very general, the fiber $f^{-1}(f(P_i))$ computes $\varepsilon(L; P_1, \ldots, P_r)$ for arbitrary $i = 1, \ldots, r$.*
*Moreover the factor $\sqrt{\frac{r-1}{r}}$ is optimal for every $r$.*

*Proof.* Let $P_1, \ldots, P_r \in X$ be very general. Since $\varepsilon(L; P_1, \ldots, P_r)$ is not maximal, there exists a Seshadri curve $C_{P_1, \ldots, P_r}$. Moving the points around we obtain a non-trivial family $C_t = C_{(P_1)_t, \ldots, (P_r)_t}$ of such curves. Let $m_1 \geq \ldots \geq m_r$ be non-negative integers such that $\text{mult}_{(P_i)_t} C_t = m_i$ for the general member $C_t$ of the family.

We proceed by induction on $r$ and begin with $r = 2$ (note that our assertion is empty for $r = 1$ and we cannot use the Hwang-Keum result as the first step of the induction).

First we assume that $m_1 \geq m_2 \geq 1$. From Lemma 2.3 we obtain

$$(m_1^2 + m_2^2 - m_2) \cdot L^2 \leq (C_t)^2 \cdot L^2.$$

On the other hand, by our numerical assumptions we have

$$(L.C_t)^2 < (m_1 + m_2)^2 \cdot \frac{1}{4} \cdot L^2.$$

Both inequalities can be written in one line thanks to the Hodge Index Theorem:

$$m_1^2 + m_2^2 - m_2 < \frac{1}{4}(m_1 + m_2)^2.$$

This is equivalent to

$$2(m_1^2 + m_2^2) + (m_1 - m_2)^2 < 4m_2,$$

which is false as $m_1 \geq m_2 \geq 1$.

If $m_2 = 0$, then by the assumption of the Theorem

$$\frac{L.C_t}{m_1} < \sqrt{\frac{1}{4}L^2} < \sqrt{\frac{3}{4}L^2} = \sqrt{\frac{3}{4}}\varepsilon_{\text{upper}}(L;1)$$

and in this case our assertion follows from Theorem 3.1.

For the induction step assume that the number of points $r$ is at least 3. There are the following possibilities:
(a) $m_1 \geq \ldots \geq m_r \geq 1$ and $m_1 \geq 2$ or
(b) $m_1 = \ldots = m_r = 1$ or
(c) $m_r = 0$.

In case (a) the Hodge Index Theorem together with Lemma 2.3 give:

$$\frac{\sum m_i^2 - m_r}{(\sum m_i)^2}L^2 \leq \frac{L^2 \cdot (C_t)^2}{(\sum m_i)^2} \leq \frac{(L.C_t)^2}{(\sum m_i)^2} < \frac{r-1}{r^2}L^2.$$

Hence by Lemma 2.4 we obtain

$$\sum m_i^2 - m_r < \frac{r-1}{r^2}\left(\sum m_i\right)^2 < \frac{(r-1)(r+1)}{r^2}\left(\sum m_i^2 - m_r\right),$$

a contradiction.

Case (b) is also immediately excluded as $(C_t)^2 \geq r - 1$ by Lemma 2.3 and thus

$$\frac{L.C_t}{\sum m_i} = \frac{L.C_t}{r} \geq \frac{\sqrt{r-1}}{r}\sqrt{L^2}$$

by Hodge Index Theorem contradicting our assumption (1).

In the last case (c) we have

$$\frac{L.C_t}{\sum_{i=1}^{r-1} m_i} = \frac{L.C_t}{\sum_{i=1}^{r} m_i} < \sqrt{\frac{r-1}{r^2}L^2} < \sqrt{\frac{r-2}{(r-1)^2}L^2},$$

where the first inequality is just our assumption (1) and the second holds as $r \geq 3$. This shows that the assumptions of our Theorem are satisfied for $r - 1$ and we conclude by induction. $\qquad\square$

6

The following example shows that our bound is optimal.

**Example 4.2** *Let $X = \mathbb{P}^2$, let $L = \mathcal{O}_{\mathbb{P}^2}(1)$ and let $r = 2$. Then the line through two given points $P_1, P_2$ computes $\varepsilon(L; P_1, P_2) = \frac{1}{2} = \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\text{upper}}(L; 2)$ and there is no fibration on $\mathbb{P}^2$.*

*More generally, let $r$ be given and let $X$ be a rational normal scroll in $\mathbb{P}^r$ and $L = \mathcal{O}_X(1)$. The scroll is of course fibered but the curves in the ruling are not the Seshadri curves. To see this let $P_1, \ldots, P_r \in X$ be points in general position. Then obviously for a fiber $F$ of the ruling passing through one or more points of the set $P_1, \ldots, P_r$ we have*

$$\frac{L.F}{\sum \text{mult}_{P_i} F} = 1.$$

*On the other hand $r$ points span a hyperplane in $P^r$ i.e. there is a curve $C \in |L|$ passing through all of them with Seshadri quotient*

$$\frac{L.C}{\sum \text{mult}_{P_i} C} = \frac{r-1}{r} = \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\text{upper}}(L; r) < 1.$$

*Now, Bertini Theorem implies that $C$ is irreducible for general $P_1, \ldots, P_r$. Let $D \neq C$ be an irreducible curve passing through at least one of the points $P_1, \ldots, P_r$. Then*

$$L.D = L.C \geq \sum_{i=1}^{r} \text{mult}_{P_i} D \cdot \text{mult}_{P_i} C \geq \sum_{i=1}^{r} \text{mult}_{P_i} D.$$

*This shows that the hyperplane section is the only Seshadri curve on $X$ (In fact this is always the case if there is a Seshadri curve in $|L|$ itself). So $X$ is not fibered by the Seshadri curves in this case.*

Our results deal with situations when Seshadri constants are relatively small related to the upper bound. In fact it is conjectured that Seshadri constants at sufficiently many points are always maximal. Below we formulate this conjecture more exactly, it interpolates on the well known Nagata conjecture, see [9] for an effective statement, background and equivalent formulations.

**Conjecture 4.3 (Nagata-Biran)** *Let $X$ be a smooth projective variety and $L$ an ample line bundle on $X$. Then there exists a number $r_0$ (depending on $X$ and $L$) such that for all $r \geq r_0$*

$$\varepsilon(L; r) = \varepsilon_{\text{upper}}(L; r).$$

Theorem 4.1 can be viewed as an asymptotic confirmation of the above conjecture. A similar result was obtained with different methods by Harbourne [5].

**Corollary 4.4** *If a surface $X$ admits no fibration over a curve (e.g. a general surface of general type), then*

$$\varepsilon(L; r) \geq \sqrt{\frac{r-1}{r}} \cdot \varepsilon_{\text{upper}}(L; r).$$

*In particular the Nagata-Biran conjecture holds on $X$ asymptotically.*

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